C3 Exercise 7A (trigonometric addition formulae)

Using

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

→ INSERT PROOFS & EXPLANATIONS

1 find a counter example to the incorrect assertion that

$$\sin(A+B) \equiv \sin A + \sin B$$

probably best to choose $A = B = 45^\circ$

$$\sin(A+B) = \sin 90^\circ = 1$$

$$\sin A + \sin B = \sin 45 + \sin 45 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$\therefore \sin(A+B) \neq \sin A + \sin B$ when $A = B = 45^\circ$.

$A = 60^\circ$ $B = 30^\circ$ also works well

2 Use the expansion of $\cos(A - B)$ with $A = B = \theta$ show that $\sin^2\theta + \cos^2\theta \equiv 1$

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

$$\Rightarrow \cos(\theta - \theta) \equiv \cos\theta\cos\theta + \sin\theta\sin\theta$$

$$\Rightarrow \cos 0^\circ \equiv \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \sin^2\theta + \cos^2\theta \equiv 1$$

3a Use the expansion of $\sin(A-B)$ to show that $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\Rightarrow \sin\left(\frac{\pi}{2}-\theta\right) \equiv \sin\left(\frac{\pi}{2}\right)\cos\theta - \cos\left(\frac{\pi}{2}\right)\sin\theta$$

now $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$

$$\Rightarrow \sin\left(\frac{\pi}{2}-\theta\right) \equiv 1\cos\theta - 0\sin\theta$$

$$\Rightarrow \sin\left(\frac{\pi}{2}-\theta\right) \equiv \cos\theta$$

(The easy way to prove this is to consider the triangle instead)

(Also have a picture of the graphs in mind... tie the ideas together)

3b Use the expansion of $\cos(A-B)$ to show that $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$

put $A = \pi/2$ and $B = \theta$

$\Rightarrow \cos\left(\frac{\pi}{2}-\theta\right) \equiv \cos\left(\frac{\pi}{2}\right)\cos\theta + \sin\left(\frac{\pi}{2}\right)\sin\theta$

now $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow \cos\left(\frac{\pi}{2}-\theta\right) \equiv 0\cos\theta + 1\sin\theta$

$\Rightarrow \cos\left(\frac{\pi}{2}-\theta\right) \equiv \sin\theta$

4 Express as a single sine, cosine or tangent

a $\sin 15^\circ \cos 20^\circ + \cos 15^\circ \sin 20^\circ \equiv \sin(15+20)$

$\equiv \sin 35^\circ$

b $\sin 58^\circ \cos 23^\circ - \cos 58^\circ \sin 23^\circ \equiv \sin(58-23)$

$\equiv \sin 35^\circ$

c $\cos 130^\circ \cos 80^\circ - \sin 130^\circ \sin 80^\circ \equiv \cos(130+80)$

$\equiv \cos 210^\circ$

4d $\dfrac{\tan 76^\circ - \tan 45^\circ}{1 + \tan 76^\circ \tan 45^\circ} \equiv \tan(76 - 45)$

$\equiv \tan 31^\circ$

e $\cos 2\theta \cos\theta + \sin 2\theta \sin\theta \equiv \cos(2\theta - \theta)$

$\equiv \cos\theta$

f $\cos 4\theta \cos 3\theta - \sin 4\theta \sin 3\theta \equiv \cos(4\theta + 3\theta)$

$\equiv \cos 7\theta$

4g $\sin \frac{1}{2}\theta \cos 2\frac{1}{2}\theta + \cos\frac{1}{2}\theta \sin 2\frac{1}{2}\theta \equiv \sin\left(\frac{1}{2}\theta + 2\frac{1}{2}\theta\right)$

$\equiv \sin 3\theta$

h $\dfrac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta} \equiv \tan(2\theta + 3\theta)$

$\equiv \tan 5\theta$

i $\sin(A+B)\cos B - \cos(A+B)\sin B \equiv \sin(A+B-B)$

$\equiv \sin A$

4j $\cos\left(\frac{3x+2y}{2}\right)\cos\left(\frac{3x-2y}{2}\right) - \sin\left(\frac{3x+2y}{2}\right)\sin\left(\frac{3x-2y}{2}\right)$

$$\equiv \cos\left\{\left(\frac{3x+2y}{2}\right) + \left(\frac{3x-2y}{2}\right)\right\}$$

$$\equiv \cos 3x$$

5a Calculate, without using your calculator, the exact value of

$$\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ \equiv \sin 90 \equiv 1$$

5b $\cos 110^\circ \cos 20 + \sin 110 \sin 20^\circ$

$$= \cos(110-20)$$

$$= \cos 90$$

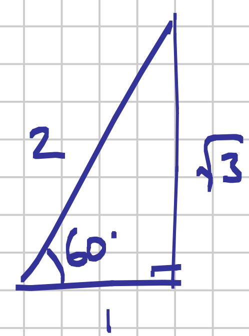
$$= 0$$

5c $\sin 33^\circ \cos 27^\circ + \cos 33 \sin 27^\circ$

$= \sin(33 + 27)$

$= \sin 60^\circ$

$= \frac{\sqrt{3}}{2}$

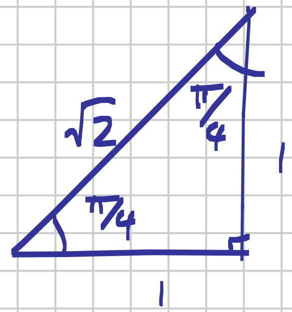


5d $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$

$$= \cos\left(\frac{\pi}{8}+\frac{\pi}{8}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

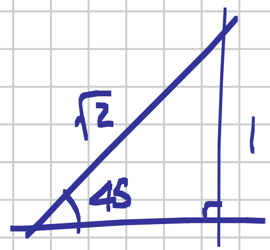


5e $\sin 60^\circ \cos 15^\circ - \cos 60^\circ \sin 15^\circ$

$= \sin(60 - 15)$

$= \sin 45^\circ$

$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$



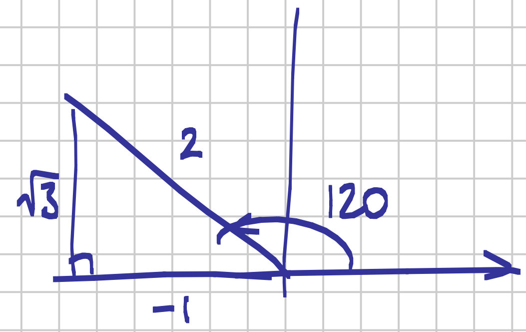
5f $\cos 70\left(\cos 50^\circ - \tan 70^\circ \sin 50^\circ\right)$

$= \cos 50 \cos 70 - \sin 50 \sin 70$

$= \cos(50 + 70)$

$= \cos 120^\circ$

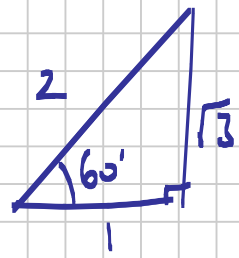
$= -\frac{1}{2}$



5g

$$\frac{\tan 45 + \tan 15}{1 - \tan 45 \cdot \tan 15} = \tan(45 + 15) = \tan 60^\circ$$

$$= \sqrt{3}$$

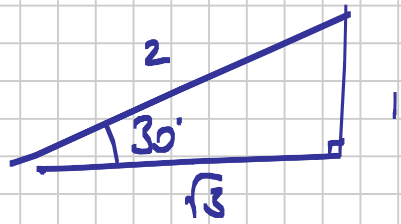


5h

$$\frac{1 - \tan 15^\circ}{1 + \tan 15^\circ}$$

hint: $\tan 45^\circ = 1$

$$= \frac{1 - \tan 45 \tan 15}{1 + \tan 45 \tan 15}$$



$$= \tan(45 - 15) = \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

5i

$$\frac{\tan\left(\frac{7\pi}{12}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(\frac{7\pi}{12}\right)\tan\left(\frac{\pi}{3}\right)} = \tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right)$$

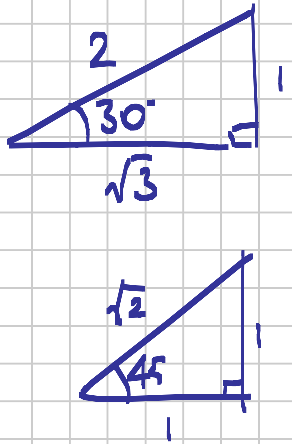
$$= \tan\left(\frac{7\pi}{12} - \frac{4\pi}{12}\right)$$

$$= \tan\left(\frac{3\pi}{12}\right)$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$= 1$$

5j $\sqrt{3}\cos 15^\circ - \sin 15^\circ \equiv 2\left[\frac{\sqrt{3}}{2}\cos 15 - \frac{1}{2}\sin 15\right]$



$\equiv 2\left[\cos 30^\circ \cos 15 - \sin 30 \sin 15\right]$

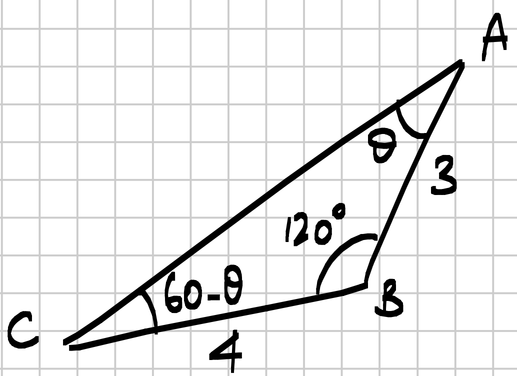
$\equiv 2\cos(30+15)$

$\equiv 2\cos 45^\circ$

$\equiv 2 \times \frac{\sqrt{2}}{2} \equiv \sqrt{2}$

6 $\triangle ABC$. $AB = 3cm$, $BC = 4cm$, $\angle ABC = 120^\circ$ $\angle BAC = \theta^\circ$

a

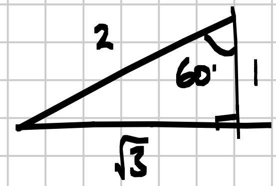


$\angle ACB = 180 - 120 - \theta$

$\angle ACB = 60 - \theta$

6b using the sine rule, or otherwise show that $\tan\theta = \frac{2\sqrt{3}}{5}$

$$\frac{\sin(60-\theta)}{3} = \frac{\sin\theta}{4}$$



$\cos 60 = \frac{1}{2}$

$\sin 60 = \sqrt{3}/2$

$$\frac{\sin 60\cos\theta - \cos 60\sin\theta}{3} = \frac{\sin\theta}{4}$$

$$4\sin 60\cos\theta - 4\cos 60\sin\theta = 3\sin\theta$$

$$2\sqrt{3}\cos\theta - 2\sin\theta = 3\sin\theta$$

$$2\sqrt{3}\cos\theta = 5\sin\theta$$

$$\frac{2\sqrt{3}}{5} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

7a prove that $\sin(A+60) + \sin(A-60) \equiv \sin A$

$$\sin(A+60^\circ) + \sin(A-60) \equiv \sin A\cos 60 + \cancel{\cos A \sin 60} + \sin A\cos 60 - \cancel{\cos A\sin 60}$$

$$\equiv 2\sin A\cos 60$$

... but $\cos 60^\circ = \frac{1}{2}$ so...

$$\equiv \sin A. \quad \square$$

7b Prove $\dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} \equiv \dfrac{\cos(A+B)}{\sin B \cos B}$.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

so $$RHS = \frac{\cos A \cancel{\cos B}}{\sin B \cancel{\cos B}} - \frac{\sin A \cancel{\sin B}}{\cancel{\sin B} \cos B}$$

$$= \frac{\cos A}{\sin B} - \frac{\sin A}{\cos B}. \quad \square$$

7c Prove that $\dfrac{\sin(x+y)}{\cos x \cos y} \equiv \tan x + \tan y$

$$\text{LHS} \equiv \frac{\sin(x+y)}{\cos x \cos y} \equiv \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}$$

$$= \frac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cos y}$$

$$\equiv \tan x + \tan y \,. \square$$

7d Prove $\dfrac{\cos(x+y)}{\sin x \sin y} + 1 \equiv \cot x \cot y$

$$\text{LHS} \equiv \frac{\cos x \cos y - \sin x \sin y}{\sin x \sin y} + 1$$

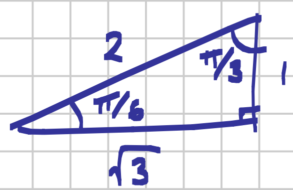
$$\equiv \cot x \cot y - 1 + 1$$

$$\equiv \cot x \cot y \equiv \text{RHS}. \square$$

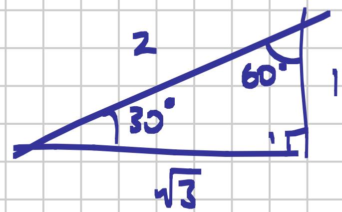
7e Prove $\cos\left(\theta + \frac{\pi}{3}\right) + \sqrt{3}\sin\theta = \sin\left(\theta + \frac{\pi}{6}\right)$

start with looking at $\frac{\pi}{3}$ and $\frac{\pi}{6}$

Notice



$\equiv$



So

$\tan\frac{\pi}{6} = \cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$ $\tan\frac{\pi}{3} = \cot\frac{\pi}{6} = \sqrt{3}$

$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$ $\sin\frac{\pi}{3} = \cos\pi/6 = \frac{\sqrt{3}}{2}$

Now

$LHS \equiv \cos\left(\theta + \frac{\pi}{3}\right) + \sqrt{3}\sin\theta$

$$\equiv \cos\theta \cos\frac{\pi}{3} - \sin\theta \sin\frac{\pi}{3} + \sqrt{3}\sin\theta$$

$$\equiv \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta + \sqrt{3}\sin\theta$$

$$\equiv \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$

$$\equiv \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta$$

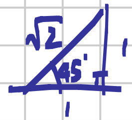
$$\equiv \sin\left(\theta + \frac{\pi}{6}\right). = RHS \;\square.$$

7f Prove $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

$$\cot(A+B) \equiv \frac{\cos(A+B)}{\sin(A+B)}$$

$$\equiv \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B} \times \frac{\frac{1}{\sin A \sin B}}{\frac{1}{\sin A \sin B}}$$

$$\equiv \frac{\cot A \cot B - 1}{\cot B + \cot A}$$



$$\Rightarrow \quad \sin 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

7g Prove $\sin^2(45+\theta)^\circ + \sin^2(45-\theta)^2 \equiv 1$

$$LHS \equiv (\sin 45 \cos\theta + \cos 45 \sin\theta)^2 + (\sin 45 \cos\theta - \cos 45 \sin\theta)^2$$

$$\equiv \left(\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta\right)^2 + \left(\frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta\right)^2$$

$$\equiv \frac{1}{2}\cos^2\theta + \sqrt{2}\sin\theta\cos\theta + \frac{1}{2}\sin^2\theta$$

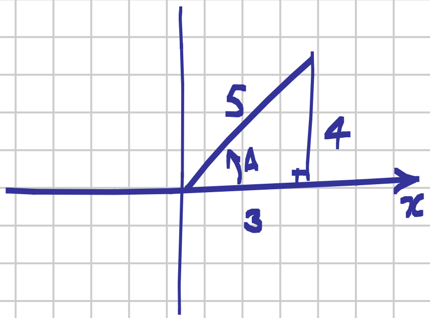
$$+ \frac{1}{2}\cos^2\theta - \sqrt{2}\sin\theta\cos\theta + \frac{1}{2}\sin^2\theta$$

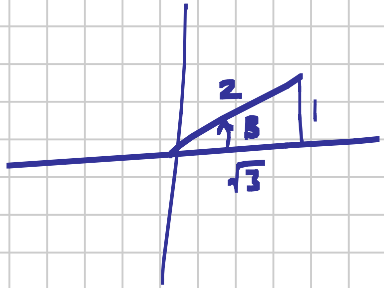
$$\equiv \cos^2\theta + \sin^2\theta \equiv 1 = RHS. \ \square.$$

7h Prove $\cos(A+B)\cos(A-B) \equiv \cos^2 A - \sin^2 B$

$$\begin{aligned}
LHS &\equiv \cos(A+B)\cos(A-B) \\
&\equiv (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\
&\equiv \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\
&\equiv
\end{aligned}$$

8a $\sin A = \frac{4}{5}$ $\sin B = \frac{1}{2}$





$\sin A = 4/5$
$\cos A = 3/5$

$\sin B = \frac{1}{2}$
$\cos B = \sqrt{3}/2$

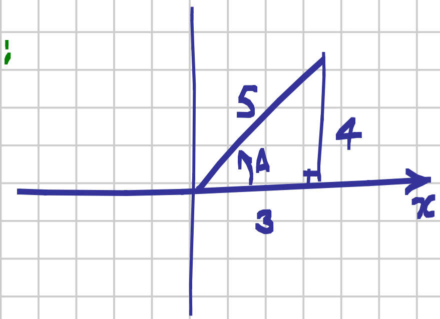
a find $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$

$$\equiv \frac{4}{5} \times \frac{\sqrt{3}}{2} + \frac{3}{5} \times \frac{1}{2}$$

$$= \frac{4\sqrt{3} + 3}{10}$$

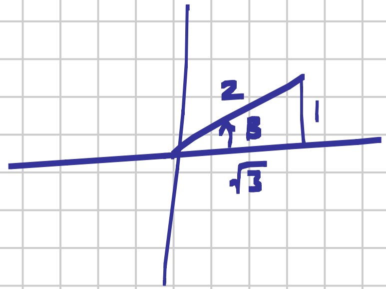
8b $\cos(A-B) = \cos A \cos B + \sin A \sin B$

from 8a) we had:



$\sin A = 4/5$

$\cos A = 3/5$



$\sin B = \frac{1}{2}$

$\cos B = \sqrt{3}/2$

$$\Rightarrow \cos(A-B) = \left(\frac{3}{5}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{2}\right)$$

$$= \frac{3\sqrt{3}}{10} + \frac{4}{10}$$

$$= \frac{4+3\sqrt{3}}{10}$$

8c

$$\sec(A-B) = \frac{1}{\cos(A-B)}$$

$$= \frac{1}{\left(\frac{4+3\sqrt{3}}{10}\right)}$$

$$= \frac{10}{4+3\sqrt{3}}$$

$$= \frac{10}{4+3\sqrt{3}} \times \frac{4-3\sqrt{3}}{4-3\sqrt{3}}$$

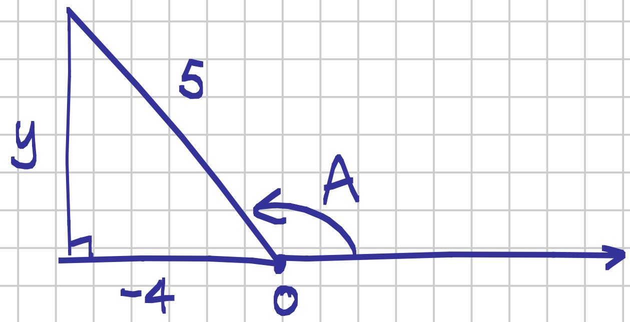
$$= \frac{10(4-3\sqrt{3})}{16-27}$$

$$= \frac{10(4-3\sqrt{3})}{-11} = \frac{10(3\sqrt{3}-4)}{11}.$$

You should automatically be inclined to rationalise the denominator. See [C1] Exercise 1H.

9. Given that $\cos A = -\frac{4}{5}$ and A is an obtuse angle measured in radians find the exact value of

a $\sin A$



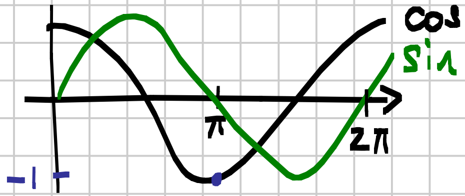
$(-4)^2 + y^2 = 5^2$

$\Rightarrow y^2 = 9$

$y = 3$

$\sin A = \frac{3}{5}$

9b find $\cos(\pi + A)$



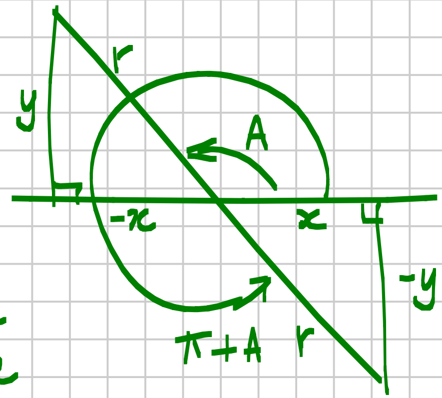
$$\cos(\pi + A) = \cos\pi \cos A - \sin\pi \sin A$$

$$= -\cos A - 0$$

$$= -\cos A$$

$$= \frac{4}{5}$$

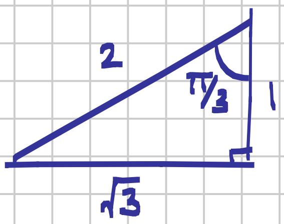
Think about the picture:



angle $\pi + A$ changes $-x$ into x
and (of course) keeps r positive
so $\frac{-4}{5}$ becomes $\frac{4}{5}$.

9c $\sin\left(\frac{\pi}{3} + A\right) = \sin\frac{\pi}{3}\cos A + \cos\frac{\pi}{3}\sin A$

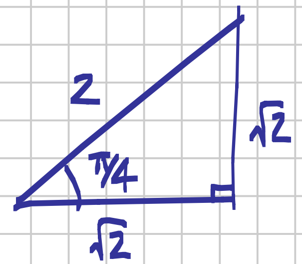
standard triangle: $\Rightarrow$ $\sin\left(\frac{\pi}{3} + A\right) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{-4}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$



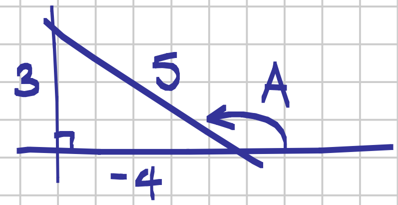
$\sin\left(\frac{\pi}{3} + A\right) = \frac{3 - 4\sqrt{3}}{10}$

9d $\tan\left(\frac{\pi}{4}+A\right) = \dfrac{\tan\frac{\pi}{4} + \tan A}{1 - \tan\frac{\pi}{4}\tan A}$

now $\tan\left(\frac{\pi}{4}\right) = 1$ because



and $\tan A = -3/4$ because



so $\tan\left(\frac{\pi}{4}+A\right) = \dfrac{1 + -3/4}{1-(1)(-3/4)}$

$= 1/4 \div \left(1 + 3/4\right) = \frac{1}{4} \div \frac{7}{4} = \frac{1}{\cancel{4}} \times \frac{\cancel{4}}{7}$

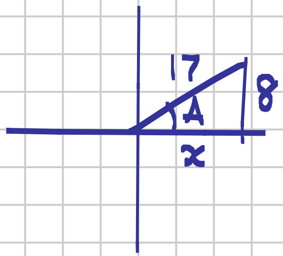
$\Rightarrow \tan\left(\frac{\pi}{4}+A\right) = \frac{1}{7}$

10 Given that $\sin A = \frac{8}{17}$, where A is acute

and $\cos B = -\frac{4}{5}$, where B is obtuse

calculate the exact value of ...

(a) $\sin(A - B)$

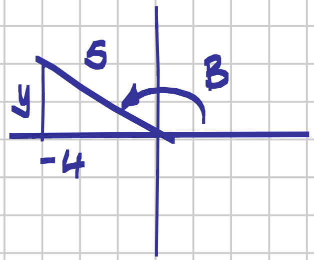


$x^2 + 8^2 = 17^2$

$x^2 + 64 = 289$

$x^2 = 225$

$x = 15$



$y^2 + (-4)^2 = 5^2$

$\Rightarrow y = 3$

$\Rightarrow \cos A = \frac{15}{17}$ given $\sin A = \frac{8}{17}$

and $\tan A = \frac{8}{15}$

$\Rightarrow \sin B = \frac{3}{5}$ given $\cos B = \frac{-4}{5}$

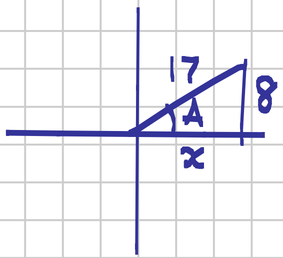
and $\tan B = \frac{-3}{4}$

Now $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\Rightarrow \sin(A - B) = \left(\frac{8}{17}\right)\left(\frac{-4}{5}\right) - \left(\frac{15}{17}\right)\left(\frac{3}{5}\right) = \frac{-32 - 45}{85} = -\frac{77}{85}$

10b $\cos(A-B) = \cos A\cos B + \sin A \sin B$

We had:



$x^2 + 8^2 = 17^2$

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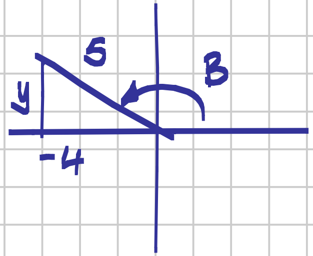
$x^2 = 225$

$x = 15$



$y^2 + (-4)^2 = 5^2$

$\Rightarrow y = 3$

$\Rightarrow \cos A = \frac{15}{17}$ given $\sin A = \frac{8}{17}$

and $\tan A = \frac{8}{15}$

$\Rightarrow \sin B = \frac{3}{5}$ given $\cos B = \frac{-4}{5}$

and $\tan B = \frac{-3}{4}$

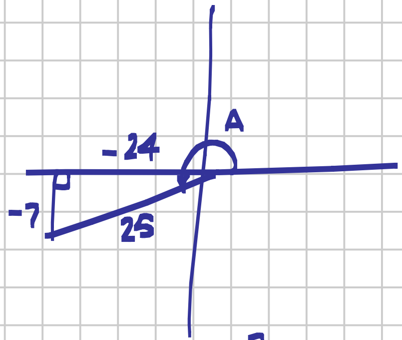
$$\Rightarrow \cos(A-B) = \left(\frac{15}{17}\right)\left(\frac{-4}{5}\right) + \left(\frac{8}{17}\right)\left(\frac{3}{5}\right) = \frac{-60+24}{85} = -\frac{36}{85}$$

10c $\cot(A-B) = \dfrac{1}{\tan(A-B)} = \dfrac{1+\tan A\tan B}{\tan A - \tan B}$

$$\cot(A-B) = \frac{1+\left(\frac{8}{15}\right)\left(\frac{-3}{4}\right)}{\frac{8}{15} - \frac{-3}{4}} = \left(\frac{60}{60} - \frac{24}{60}\right) \div \left(\frac{32}{60} + \frac{45}{60}\right) = \left(\frac{36}{60}\right) \div \left(\frac{77}{60}\right)$$

$$= \frac{36}{77}$$

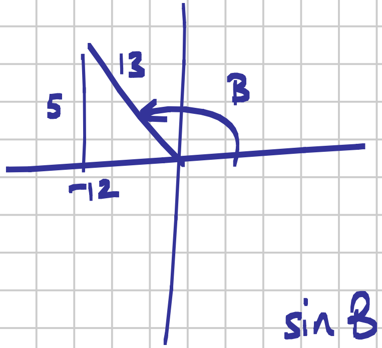
11 Given $\tan A = \frac{7}{24}$ A is reflex $\sin B = \frac{5}{13}$ B obtuse



$\sin A = {}^{-7}/_{25}$

$\cos A = {}^{-24}/_{25}$

$\tan A = \frac{-7}{-24} = {}^{7}/_{24}$



$\sin B = {}^{5}/_{13}$

$\cos B = {}^{-12}/_{13}$

$\tan B = \frac{5}{-12} = -\frac{5}{12}$

11b

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\frac{7}{24} - \frac{-5}{12}}{1 + \left(\frac{7}{24}\right)\left(\frac{-5}{12}\right)}$$

$$= \frac{\frac{17}{24}}{\frac{253}{24 \times 12}}$$

$$= \frac{34}{253}.$$

13

F

15

16

18 Given that $3\sin(x-y) - \sin(x+y) = 0$

Show that $\tan x = 2\tan y$.

$$3\sin(x-y) \equiv 3(\sin x \cos y - \cos x \sin y)$$

$$\sin(x+y) \equiv \sin x \cos y + \cos x \sin y$$

$$\Rightarrow \quad 3\sin x \cos y - 3\cos x \sin y - \sin x \cos y - \cos x \sin y = 0$$

$$\Rightarrow \quad 2\sin x \cos y - 4\cos x \sin y = 0$$

$$2\sin x \cos y = 4\cos x \sin y$$

$$\tan x \cos y = 2\sin y$$

$$\tan x = 2\tan y$$

b solve $3\sin(x-45) - \sin(x+45) = 0$ for $0 \leq \theta \leq 360$

19 Given that

$$\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y) \qquad (1)$$

find $\tan(x+y)$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$(1) \Rightarrow$
$$\frac{\sin x}{\cos x} = \frac{2\cos y - \sin y}{\cos y + 2\sin y}$$

and

$(1) \Rightarrow$
$$\sin x \cos y + 2\sin x \sin y = 2\cos x \cos y - \cos x \sin y$$

$$\Rightarrow \quad \cos x \sin y + 2 \sin x \sin y = 2\cos x \cos y - \sin x \cos y$$

$$\Rightarrow \quad \sin y\,(\cos x + 2\sin x) = \cos y\,(2\cos x - \sin x)$$

$$\Rightarrow \quad \tan x = \frac{2\cos x - \sin x}{\cos x + 2\sin x}$$

This is making progress, but it's gonna get ugly.

Start over: try $\tan(x+y) = \dfrac{\sin(x+y)}{\cos(x+y)}$

now $\sin(x+y) = \sin x \cos y + \cos x \sin y$

19 ctd

and $\cos(x+y) = \cos x \cos y - \sin x \sin y$

1 $\Rightarrow \sin x \cos y + 2 \sin x \sin y - 2 \cos x \cos y + \cos x \sin y = 0$

$\Rightarrow \sin(x+y) - 2\cos(x+y) = 0$

$\Rightarrow \sin(x+y) = 2\cos(x+y)$

$\Rightarrow \tan(x+y) = 2.$

20 Given that $\tan(x-y) = 3$ express $\tan y$ in terms of $\tan x$.

$$\tan(x-y) \equiv \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\Rightarrow \quad \tan x - \tan y = 3(1 + \tan x \tan y)$$

$$\tan x - \tan y = 3 + 3\tan x \tan y$$

rearrange to put all the $\tan y$ terms on one side

$$\tan x - 3 = 3\tan x \tan y + \tan y$$

$$\tan x - 3 = \tan y\,(3\tan x + 1)$$

factorise!

$$\tan y = \frac{\tan x - 3}{3\tan x + 1}$$